Interference Channels with CoMP: Degrees of Freedom, Message Assignment, and Fractional Reuse

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Abstract

Coordinated Multi-Point (CoMP) transmission is an infrastructural enhancement under consideration for next generation wireless networks. In this work, the capacity gain achieved through CoMP is studied in various models of wireless networks that have practical significance. The capacity gain is analyzed through the degrees of freedom (DoF) criterion. The DoF available for communication provides an analytically tractable way to characterize the capacity of interference channels. The considered channel model has K transmitter/receiver pairs, and each receiver is interested in one unique message from a set of K independent messages. Each message can available at more than one transmitter. The maximum number of transmitters at which each message can be available, is defined as the *cooperation order* M. For fully connected interference channels, it is shown that the asymptotic per user DoF as K goes to infinity, remains at $\frac{1}{2}$ as M is increased from 1 to 2. Furthermore, the same negative result holds for all $M \geq 2$ for any message assignment that satisfies a *local cooperation* constraint. On the other hand, when the assumption of full connectivity is relaxed to *local connectivity*, and each transmitter is connected only to its own receiver as well as L neighboring receivers, it is shown that local cooperation is optimal. The asymptotic per user DoF is shown to be at least $\max\left\{\frac{1}{2}, \frac{2M}{2M+L}\right\}$ for locally connected channels, and is shown to be $\frac{2M}{2M+1}$ for the special case of *Wyner's asymmetric model* where $L = 1$. An interesting feature of the proposed achievability scheme is that it relies on simple zero-forcing transmit beams and does not require symbol extensions. Also, to achieve the optimal per user DoF for Wyner's model, messages are assigned to transmitters in an asymmetric fashion unlike traditional assignments where message i has to be available at transmitter i. It is also worth noting that some receivers have to be inactive, and *fractional reuse* is needed to achieve equal DoF for all users.

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I. INTRODUCTION

In the past decade, there has been a significant growth in the usage of wireless networks, and in particular, cellular networks, because of the increased data demands. This has been the driver of recent research for new ways of managing interference in wireless networks.

Due to the superposition and broadcast properties of the wireless medium, interfering signals pose a significant limitation to the rate of communication of users in a wireless network. Hence, it is of interest to understand the fundamental limits of communication in interference channels and to capture the effect of interference on optimal encoding and decoding schemes. The problem of finding the capacity region of even the simple $2-$user Gaussian interference channel is still an open problem. However, approximations exist in the literature, where the capacity region or the sum capacity is known in the special scenarios of strong and low interference ([3], [4], [5], [6]). Another effective approximation that simplifies the problem is to consider only the sum degrees of freedom (DoF) or the pre-log factor of the sum capacity at high signal-to-noise ratio (SNR). The DoF criterion provides an analytically tractable way to characterize the sum capacity and captures the number of interference-free sessions that can be supported in a given multi-user channel.

In [7], the DoF of the fully connected $K-$user Gaussian interference channel was shown to be upper bounded by $K/2$, i.e., the per user DoF is bounded by $1/2$. This was shown to be achievable through the interference alignment (IA) scheme in [8]. However, this achievable DoF many not be sufficient to meet the demands of wireless applications in many scenarios of practical interest, and hence, it is of interest to study ways to enhance the infrastructure of wireless networks in order to increase the rate of communication.

The considered infrastructural enhancement in this work is the deployment of a backhaul link, through which the transmitters/base stations can exchange messages that they wish to deliver in a cellular downlink session[1]. In order to model the finite capacity of the backhaul link, we impose a cooperation constraint where each message can be available at a maximum of M transmitters. We call M the *cooperation order*. The availability of each message at more than one transmitter allows for Coordinated Multi-Point (CoMP) transmission [10]. In [11], a CoMP transmission model for the fully connected K-user interference channel was considered. Each message was assumed to be available at the transmitter carrying the same index as the message as well as $M-1$ succeeding transmitters. Using an extension of the asymptotic interference alignment scheme of [8], the DoF of the channel in this setting was shown to be $\frac{K+M-1}{2}, \forall K < 10$, and it was conjectured that this expression is valid for all values of K. We note that this DoF cooperation gain beyond $K/2$ does not scale linearly with K as K goes to infinity. In other words, the asymptotic per user DoF remains $1/2$. In Section IV, we study whether there exists an assignment of messages satisfying the cooperation order constraint that enables the achievability of an asymptotic per user DoF that is strictly greater than $1/2$.

[1]The considered scenario has more practical relevance than the the cellular uplink model where base station receivers can cooperate by sharing analog signals. Nevertheless, as discussed in [9] for fully connected channels, the results obtained using linear schemes in our transmitter cooperation model can be obtained in the dual receiver cooperation model.

The assumption of full connectivity is key to the results obtained in [7], [8], [11], and in Section IV of this work. For the fully connected interference channel, interference mitigating schemes are designed to avoid the interference caused by all other transmitters in the network. However, in practice, each receiver gets most of the destructive interference from a few dominant interfering transmitters. For example, in cellular networks, the number of dominant interfering transmit signals at each receiver ranges from two to seven. All the interference from the remaining transmitters may contribute to the interference floor, and the improvement obtained by including them in the dominant interferers set may not justify the corresponding overhead. For this reason, we study locally connected channels in Section V, where the channel coefficients between transmitters and receivers that lie at a distance that is greater than some threshold are approximated to equal zero.

For the locally connected channel model, we assume that each transmitter is connected to L neighboring receivers as well as the receiver carrying its own index, $\left\lfloor \frac{L}{2} \right\rfloor$ preceding receivers and $\left\lceil \frac{L}{2} \right\rceil$ succeeding receivers. The special case of this model where $L = 1$ is Wyner's asymmetric model [25]. This special case was considered in [12], and it was assumed that each message is available at the transmitter carrying the same index as well as $M - 1$ succeeding transmitters. The asymptotic per user DoF was characterized under this setting as $\frac{M}{M+1}$. The achieving scheme is simple as it relies only on simple zero-forcing transmit beamforming. In Section V, we extend this result and characterize the asymptotic per user DoF for Wyner's asymmetric model as $\frac{2M}{2M+1}$ under a general cooperation order constraint. The message assignment enabling this result uses only local cooperation, that is, each message is available only at neighboring transmitters, where the size of the neighborhood does not scale linearly with the size of the network, thereby, enjoying the same advantage as the message assignment considered in [12].

A. Document Organization

The remainder of this work is organized as follows. Related work is summarized in Section I-B. The problem setup is then introduced in Section II. An informal summary of results is provided in Section III. The asymptotic per user DoF of the fully connected interference channel with CoMP transmission is studied in Section IV. The locally connected channel model is considered in Section V. The introduced results are then discussed in Section VI, and the paper is concluded with some final remarks in Section VII.

B. Related Work

Many existing works studying interference networks with cooperating transmitters use the term *cognitive radios* (e.g. [13], [14], [15], [16], [17]). In another body of work, unlike the considered setting where we assume that transmitters cooperate by sharing complete messages, cooperation through sharing partial message information that is considered as side information is studied (see e.g., [18]). In [19] and [20], the transmitters are allowed to cooperate through noise free bit pipes or over the air, respectively.

Communication scenarios with cooperating multiple antenna transmitters have been considered in [21] and [22] under the umbrella of the x-channel. However, in the x-channel, mutually exclusive parts of each message are given to different transmitters. This is extended in [23] to allow each part of each message to be available at more than

one transmitter, and in [18], the MIMO x-channel is studied in the setting where transmitters share further side information.

Finally, it is worth noting that in the considered setting, we implicitly assume the coordinated design of the transmit beams between all transmitters. This kind of coordination is also referred to in the literature as *transmitter cooperation*, even without the sharing of messages (see e.g. [24]).

II. Problem Setup

We use the standard model for the $K-$user interference channel with single-antenna transmitters and receivers.

$$Y_i(t) = \sum_{j=1}^{K} H_{ij}(t) X_j(t) + Z_i(t) \tag{1}$$

where t is the time index, $X_j(t)$ is the transmitted signal of transmitter j, $Y_i(t)$ is the received signal at receiver i, $Z_i(t)$ is the zero mean unit variance Gaussian noise at receiver i, and $H_{ij}(t)$ is the channel coefficient from transmitter j to receiver i over the time slot t. We remove the time index in the rest of the paper for brevity unless it is needed.

We use $[K]$ to denote the set $\{1, 2, \ldots, K\}$. For any set $\mathcal{A} \subseteq [K]$, we define the complement set $\bar{\mathcal{A}} = \{i : i \in [K], i \notin \mathcal{A}\}$. For each $i \in [K]$, let W_i be the message intended for receiver i. We use the abbreviations $W_{\mathcal{A}}$, $X_{\mathcal{A}}$, $Y_{\mathcal{A}}$, and $Z_{\mathcal{A}}$ to denote the sets $\{W_i, i \in \mathcal{A}\}$, $\{X_i, i \in \mathcal{A}\}$, $\{Y_i, i \in \mathcal{A}\}$, and $\{Z_i, i \in \mathcal{A}\}$, respectively.

A. Channel Model

We consider two different channel models in the sequel. First, we consider a fully connected interference channel where all channel coefficients are drawn independently from a continuous distribution. We next consider a locally connected channel model where channel coefficients between well separated nodes are approximated to be identically zero, the locally connected channel model is a function of the number of interferers L. We assume the following channel model:

$$H_{ij} \text{ is not identically } 0 \text{ if and only if } j \in \left[i - \left\lceil \frac{L}{2} \right\rceil, i + \left\lfloor \frac{L}{2} \right\rfloor \right] \tag{2}$$

and all channel coefficients that are not identically zero are drawn independently from a continuous distribution. We note that for values of $L = 1$ and $L = 2$, the locally connected channel reduces to the commonly known Wyner's asymmetric and symmetric linear models, respectively [25].

B. Cooperation Model

For each $i \in [K]$, let $\mathcal{T}_i \subseteq [K]$ be the transmit set of receiver i, i.e., those transmitters with the knowledge of W_i. The transmitters in $\mathcal{T}_i$ cooperatively transmit the message W_i to the receiver i. The messages $\{W_i\}$ are assumed to be independent of each other. The *cooperation order* M is defined to be the maximum transmit set size:

$$M = \max_i |\mathcal{T}_i|. \tag{3}$$

For any set $\mathcal{A} \subseteq [K]$, we define $C_{\mathcal{A}}$ as the set of messages carried by transmitters with indices in $\mathcal{A}$, i.e., the set $\{i : \mathcal{T}_i \cap \mathcal{A} \neq \phi\}$.

C. Degrees of Freedom

Let the total power constraint across all the users be P, and let $\mathcal{W}_i$ denote the alphabet for message W_i. Then the rates $R_i(P) = \frac{\log |W_i|}{n}$ are achievable if the decoding error probabilities of all messages can be simultaneously made arbitrarily small for large enough n, and this holds for almost all channel realizations. The sum capacity $\mathcal{C}_\Sigma(P)$ is the maximum value of the sum of the achievable rates. The total number of degrees of freedom (η) is defined as $\limsup_{P\to\infty} \frac{C_\Sigma(P)}{\log P}$.

For a K-user channel, we define $\eta_F(K, M)$ as the best achievable η over all choices of transmit sets satisfying the cooperation order constraint in (3). Similarly $\eta_L(K, M)$ for a locally connected channel with L interfering signals per receiver.

In order to simplify our analysis, we define the asymptotic per user DoF $\tau_F(M)$, and $\tau_L(M)$ to measure how $\eta_F(K, M)$, and $\eta_L(K, M)$ scale with K, respectively, while all other parameters are fixed.

$$\tau_F(M) = \lim_{K\to\infty} \frac{\eta_F(K, M)}{K} \tag{4}$$

$$\tau_L(M) = \lim_{K\to\infty} \frac{\eta_L(K, M, L)}{K} \tag{5}$$

For the locally connected channel model where $L > 1$, let $x = \lfloor \frac{L}{2} \rfloor$. We silence the first x transmitters, deactivate the last x receivers, and relabel the transmit signals to obtain a ($K - x$)-user channel, where the transmitter i is connected to receivers in the set $\{Y_k : k \in \{i, i + 1, \ldots, i + L\}\}$. We note that the new channel model gives the same value of $\tau_L(M)$ as the original one, since $x = o(K)$. Unless explicitly stated otherwise, we will be using this equivalent model in the rest of the paper.

D. Message Assignment Strategy

A message assignment strategy is defined by a sequence of transmit sets $(\mathcal{T}_{i,K}), i \in [K], K \in \{1, 2, \ldots\}$. For each positive integer K and $\forall i \in [K]$, $\mathcal{T}_{i,K} \subseteq [K], |\mathcal{T}_{i,K}| \leq M$. We use message assignment strategies to define the transmit sets for a sequence of $K-$user channels. For the channel in the sequence with k users, the transmit set $\mathcal{T}_i, i \in [k]$, is the transmit set $\mathcal{T}_{i,k}$ of the message assignment strategy.

We call a message assignment strategy *optimal* for a sequence of $K-$user fully connected channels, $K \in \{1, 2, \ldots\}$, if and only if there exists a sequence of coding schemes achieving $\tau_F(M)$ using the transmit sets defined by the message assignment strategy. A similar definition applies for locally connected channels.

E. Local Cooperation

We say that a message assignment strategy satisfies the local cooperation constraint, if and only if there exists a function $r(K)$ such that $r(K) = o(K)$, and

$$\mathcal{T}_{i,K} \subseteq \{i - r(K), i - r(K) + 1, \ldots, i + r(K)\}, \forall i \in [K], \forall K \in \mathbf{Z}^+ \tag{6}$$

Let $\tau_F^{(\mathrm{loc})}(M)$ and $\tau_L^{(\mathrm{loc})}(M)$ be the maximum achievable asymptotic per user DoF $\tau_F(M)$ and $\tau_L(M)$ under the additional local cooperation constraint, respectively.

III. Informal Summary of Results

In this paper, we study the benefit of CoMP transmission via the asymptotic per user DoF. In particular, we investigate whether the asymptotic per user DoF increases by allowing CoMP transmission, i.e., by allowing a cooperation order $M > 1$, and characterize this improvement, if it exists, as a function of M.

The considered problem is completely described by two system parameters, namely, the channel connectivity and the cooperation order M. We attempt to find an answer by setting two design parameters: the message assignment strategy satisfying the cooperation order constraint, and the achievable scheme.

We know from the results in [7] and [8] that the asymptotic per user DoF of the fully connected channel is $\frac{1}{2}$ if each message is available only at the transmitter carrying the same index; it is straightforward to extend this result to the case where each message can be available at any single transmitter, i.e., the case where $M = 1$, and hence, we know that $\tau_F(M = 1) = \frac{1}{2}$. In [11], it was shown that CoMP transmission achieves a DoF gain for the fully connected channel. However, this gain does not scale linearly with the number of users K. The considered message assignment strategy in [11] is the spiral strategy where each message is assigned to the transmitter carrying the same index as well as $M - 1$ succeeding transmitters. We note that this strategy satisfies the local cooperation constraint defined in Section II-E, and show in Section IV-C that local cooperation cannot achieve an asymptotic per user DoF gain for the fully connected channel. More precisely, we show that,

$$\tau_F^{(\mathrm{loc})}(M) = \tau_F(1) = \frac{1}{2}, \forall M \tag{7}$$

Furthermore, we extend this negative conclusion in Section IV-C to all message assignments that are restricted to assign each message to at most two transmitters, i.e.,

$$\tau_F(M = 2) = \tau_F(M = 1) = \frac{1}{2} \tag{8}$$

In Section V, we illustrate how the result introduced in [12] shows that asymptotic per user DoF gains are possible for the special case of the locally connected channel where each transmitter is connected to the receiver with the same index as well as one succeeding receiver ($L = 1$). In particular, the enabling message assignment strategy is the spiral strategy that satisfies the local cooperation constraint. We then introduce in Section V-C a simple zero-forcing transmit beamforming scheme that achieves a higher asymptotic per user DoF than that shown in [12]. In particular, we show that,

$$\tau_L(M) \geq \max\left\{\frac{1}{2}, \frac{2M}{2M + L}\right\}, \forall M, L \tag{9}$$

Moreover, this lower bound is optimal if we restrict ourselves to the class of schemes that satisfy an interference avoidance constraint. We then provide an upper bound in Section V-E that completes the characterization of the asymptotic per user DoF for the case where $L = 1$, i.e., showing that,

$$\tau_1(M) = \frac{2M}{2M + 1}, \forall M \tag{10}$$

In particular, the optimal message assignment strategy for the case where $L = 1$ satisfies a local cooperation constraint. We show in Section V-D that local cooperation is optimal for all locally connected channels, thereby establishing that the negative result regarding local cooperation for the fully connected channel is due only to the assumption of full connectivity.

We note that (10) implies that the asymptotic per user DoF for Wyner's asymmetric model is strictly greater than $\frac{1}{2}$ even for the case of *no cooperation* (i.e., $M = 1$). We show however in Section V-E that this is only the case for $L = 1$, and does not hold for all other locally connected channels.

A. Proof Techniques

In Section IV-B, we provide Lemma 1. The role of this Lemma is central to the proofs of all the DoF upper bounds derived in this work for both fully and locally connected channels. In particular, its Corollary 1 implies directly all of the provided upper bounds for the fully connected channel. Moreover, Corollary 1 sheds insight on the open problem of determining whether $\tau_F(M) > \frac{1}{2}$ for $M > 2$.

It is obvious for locally connected channels that some message assignments are not *useful*. For example, for the case where $M = 1$, any assignment of a message W_i to a transmitter that is not connected to the i^{th} receiver cannot achieve a positive rate for communication of that message. To prove DoF upper bounds for locally connected channels with CoMP transmission, we use Lemma 1 together with a characterization of necessary conditions on *useful message assignments*. Useful message assignment strategies for locally connected channels are discussed in Section V-D.

IV. Fully Connected Interference Channel

In this section, we investigate whether $\tau_F(M) > \frac{1}{2}$ for $M > 1$, and message assignment strategies that may lead to a positive conclusion.

A. Prior Work

We know from [7], and [8] that the per user DoF of a fully connected interference channel without cooperation is $\frac{1}{2}$, i.e., $\tau_F(1) = \frac{1}{2}$. In [11], the following spiral message assignment strategy was considered for $M \geq 1$:
$$\mathcal{T}_{i,K} = \begin{cases} \{i, i+1, \ldots, i+M-1\}, & \forall i \in [K-(M-1)] \\ \{i, i+1, \ldots, K, 1, 2, \ldots, M-(K-i+1)\}, & \forall i \in \{K-(M-2), K-(M-2)+1, \ldots, K\}, \end{cases}$$

Using this message assignment strategy and an asymptotic interference alignment scheme, it was shown in [11, Theorem 5] that

$$\eta_F(K, M) \geq \frac{K+M-1}{2}, \forall M \leq K < 10, \tag{11}$$

and it was shown in [11] that this lower bound is within one degree of freedom from the maximum achievable DoF using the spiral message assignment strategy. However, even if $\eta_F(K, M) = \frac{K+M-1}{2}$ for all values of K, the DoF

gain due to CoMP transmission (beyond $\frac{K}{2}$) does not scale with the number of users K. Hence, the question of whether $\tau_F(M) > \frac{1}{2}$ for $M > 1$ remains open. Here, we note that the spiral message assignment strategy satisfies the local cooperation constraint and in Section IV-C, we generalize the negative conclusion of [11] to all message assignment strategies satisfying the local cooperation constraint.

B. DoF Upper Bound

In order to characterize the DoF of the channel $\tau_F(M)$, we need to consider all possible strategies for message assignments satisfying the cooperation order constraint defined in (3). Here, we provide a way to upper bound the maximum achievable DoF for each such assignment, thereby, introducing a criterion for comparing different message assignments satisfying (3) using the special cases where this bound holds tightly.

We start by stating the following auxiliary lemma for any K-user Gaussian interference channel with a DoF number of η. For any set $\mathcal{A} \subseteq [K]$, define $U_{\mathcal{A}} = \cup_{i \notin \mathcal{A}} \mathcal{T}_i$, then,

Lemma 1: If there exists a set $\mathcal{A} \subseteq [K]$ and a function f, such that $f\left(Y_{\mathcal{A}}, Z_{\mathcal{A}}, X_{\bar{U}_{\mathcal{A}}}\right) = X_{U_{\mathcal{A}}}$, then $\eta \leq |\mathcal{A}|$.

Proof: The proof is available in Appendix. Here, we provide a sketch. Recall that $Y_{\mathcal{A}} = \{Y_i, i \in \mathcal{A}\}$, $Z_{\mathcal{A}} = \{Z_i, i \in \mathcal{A}\}$, and $W_{\mathcal{A}} = \{Z_i, i \in \mathcal{A}\}$, and note that $X_{\bar{U}_{\mathcal{A}}}$ is the set of transmit signals that does not carry messages outside $W_{\mathcal{A}}$. Fix a reliable communication scheme for the $K-$user fully connected channel, and assume that there is only one centralized decoder that has access to the received signals $Y_{\mathcal{A}}$. In this case, the sum DoF is clearly bounded by $|\mathcal{A}|$, as it is the number of received signals used for decoding. We only need to show that all messages can be recovered reliably using the centralized decoder.

We can assume that the centralized decoder has access to all Gaussian noise signals since this will not affect the DoF analysis. Using $Y_{\mathcal{A}}$, the messages $W_{\mathcal{A}}$ can be recovered reliably, and hence, the signals $X_{\bar{U}_{\mathcal{A}}}$ can be reconstructed. Using $Y_{\mathcal{A}}$, $Z_{\mathcal{A}}$, and $X_{\bar{U}_{\mathcal{A}}}$, the remaining transmit signals can be reconstructed using the function f of the hypothesis. Finally, using all transmit signals and $Z_{\bar{\mathcal{A}}}$, the received signals $Y_{\bar{\mathcal{A}}}$ can be reconstructed, and the messages $W_{\bar{\mathcal{A}}}$ can then be recovered. ■

We note that Lemma 1 applies to all considered channel models. Now, we prove the following corollary for the fully connected channel. Recall that for a set of transmitter indices $\mathcal{S}$, the set $C_{\mathcal{S}}$ is the set of messages carried by transmitters in $\mathcal{S}$.

Corollary 1: For any $m, \bar{m} : m + \bar{m} \geq K$, if there exists a set $\mathcal{S}$ of indices for transmitters carrying no more than m messages, and $|\mathcal{S}| = \bar{m}$, then $\eta \leq m$, or more precisely,

$$\eta \leq \min_{\mathcal{S} \subseteq [K]} \max(|C_{\mathcal{S}}|, K - |\mathcal{S}|). \tag{12}$$

Proof: We apply Lemma 1 with the set $\mathcal{A}$ defined as follows.

Initially, set $\mathcal{A}$ as the set of indices for messages carried by transmitters with indices in $\mathcal{S}$. i.e., $\mathcal{A} = C_{\mathcal{S}}$. Now, if $|\mathcal{A}| < K - |\mathcal{S}|$, then augment the set $\mathcal{A}$ with arbitrary message indices such that $|\mathcal{A}| = K - |\mathcal{S}|$.

We now note that the above construction guarantees that $|\mathcal{A}| + |\mathcal{S}| \geq K$ and that $U_{\mathcal{A}} \subseteq \bar{\mathcal{S}}$. Hence, using Lemma 1, it suffices to show the existence of a function f such that $f(Y_{\mathcal{A}}, Z_{\mathcal{A}}, X_{\mathcal{S}}) = X_{\bar{\mathcal{S}}}$.

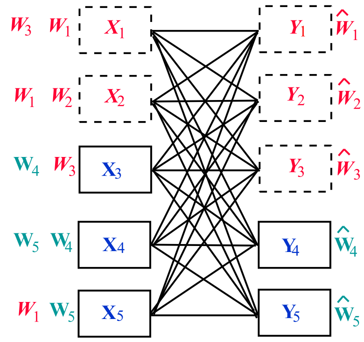

Fig. 1: Example application of Lemma 1, with $\mathcal{S} = \{1, 2\}$ and $C_{\mathcal{S}} = \{1, 2, 3\}$. Transmit signals with indices in $\mathcal{S}$, and messages as well as receive signals with indices in $C_{\mathcal{S}}$ are shown in tilted red font and dashed boxes. The DoF $\eta \leq |C_{\mathcal{S}}| = K - |\mathcal{S}| = 3$.

Given $Y_{\mathcal{A}}$, $Z_{\mathcal{A}}$, and $X_{\mathcal{S}}$, we construct the set of signal $\tilde{Y}_{\mathcal{A}}$ as follows.

$$\begin{aligned}\tilde{Y}_i &= Y_i - \left(\sum_{j \in \mathcal{S}} H_{ij} X_j + Z_i\right) \\ &= \sum_{j \in \bar{\mathcal{S}}} H_{ij} X_j, \forall i \in \mathcal{A} \end{aligned} \tag{13}$$

Since the channel is fully connected, by removing the Gaussian noise signals $Z_{\mathcal{A}}$ and transmit signals in $X_{\mathcal{S}}$ from received signals in $Y_{\mathcal{A}}$, we obtain the set of signals $\{\tilde{Y}_i : i \in \mathcal{A}\}$, which has at least $K - |\mathcal{S}| = |\bar{\mathcal{S}}|$ linear equations in the transmit signals in $X_{\bar{\mathcal{S}}}$. Moreover, since the channel coefficients are generic, those equations will be linearly independent with high probability. Hence, we can reconstruct $X_{\bar{\mathcal{S}}}$ from the linearly independent equations in (13). ■

We refer the reader to Figure 1 for an example illustration of Corollary 1.

C. Asymptotic DoF Cooperation Gain

We now use Corollary 1 to prove upper bounds on the asymptotic per user DoF $\tau_F(M)$.

In an attempt to reduce the complexity of the problem of finding an optimal message assignment strategy, we begin by considering message assignment strategies satisfying the local cooperation constraint defined in Section II-E. We now show that a scalable cooperation DoF gain cannot be achieved using local cooperation.

Theorem 1:

$$\tau_F^{(\mathrm{loc})}(M) = \frac{1}{2}, \text{ for all } M. \tag{14}$$

Proof: Fix $M \in \mathbf{Z}^+$. For any value of $K \in \mathbf{Z}^+$, we use Corollary 1 with the set $\mathcal{S} = \{1, 2, \ldots, \lceil \frac{K}{2} \rceil\}$. Note that $C_{\mathcal{S}} \subseteq \{1, 2, \ldots, \lceil \frac{K}{2} \rceil + r(K)\}$, and hence, it follows that $\eta_F(K, M) \leq \lceil \frac{K}{2} \rceil + r(K)$. Finally, $\tau_F(M) = \lim_{K \to \infty} \frac{\eta_F(K,M)}{K} \leq \frac{1}{2}$. The lower bound follows from [8] without cooperation. ■

We now investigate if it is possible for the cooperation gain to scale linearly with K for fixed M. It was shown in Theorem 1 that such a gain is not possible for message assignment strategies that satisfy the local cooperation constraint. Here, we only impose the cooperation order constraint in (3) and prove in Theorem 2 an upper bound on $\tau_F(M)$ that is tight enough for finding $\tau_F(2)$. We first prove the following auxiliary lemmas.

Lemma 2:

$$\text{There exists } i \in [K] \text{ such that } |C_{\{i\}}| \leq M. \tag{15}$$

Proof: The statement follows by the pigeonhole principle, since

$$\sum_{i=1}^{K} |C_{\{i\}}| = \sum_{i=1}^{K} |\mathcal{T}_i| \leq MK \tag{16}$$

■

Lemma 3: For $M \geq 2$, if $\exists \mathcal{A} \subset [K]$ such that $|\mathcal{A}| = n < K$, and $|C_{\mathcal{A}}| \leq (M-1)n+1$, then $\exists \mathcal{B} \subseteq [K]$ such that $|\mathcal{B}| = n+1$, and $|C_{\mathcal{B}}| \leq (M-1)(n+1)+1$.

Proof: We only consider the case where $K > (M-1)(n+1)+1$, as otherwise, the statement trivially holds. In this case, we can show that,

$$M(K - |C_{\mathcal{A}}|) < (K-n)((M-1)(n+1)+2-|C_{\mathcal{A}}|) \tag{17}$$

The proof of (17) is available in the Appendix. Note that the left hand side in the above equation is the maximum number of message instances for messages outside the set $C_{\mathcal{A}}$, i.e.,

$$\begin{aligned} \sum_{i \in [K], i \notin \mathcal{A}} |C_{\{i\}}| &\leq M(K - |C_{\mathcal{A}}|) \\ &< (K-n)((M-1)(n+1)+2-|C_{\mathcal{A}}|) \end{aligned} \tag{18}$$

Since the number of transmitters outside the set $\mathcal{A}$ is $K-n$, it follows by the pigeonhole principle that there exists a transmitter whose index is outside $\mathcal{A}$ and carries at most $(M-1)(n+1)+1-|C_{\mathcal{A}}|$ messages whose indices are outside $C_{\mathcal{A}}$. More precisely,

$$\exists i \in [K] \backslash \mathcal{A} : |C_{\{i\}} \backslash C_{\mathcal{A}}| \leq (M-1)(n+1)+1-|C_{\mathcal{A}}| \tag{19}$$

It follows that there exists a transmitter whose index is outside the set $\mathcal{A}$ and can be added to the set $\mathcal{A}$ to form the set $\mathcal{B}$ that satisfies the statement. ■

Theorem 2: For $M \geq 2$,

$$\tau_F(M) \leq \frac{M-1}{M} \tag{20}$$

Proof: We show that the following stronger statement holds.

$$\eta_F(K, M) \leq \frac{K(M-1)+M+1}{M}, \forall M \geq 2 \tag{21}$$

Assume that $n = \frac{K-1}{M}$ is an integer. We know by induction from lemmas 2 and 3 that $\exists S \subset [K]$, $|S| = n$, $|C_{\mathcal{S}}| \leq (M-1)n + 1 = \frac{K(M-1)+1}{M} = K - |\mathcal{S}|$. Now, applying Corollary 1 proves that $\eta_F(K,M) \leq \frac{K(M-1)+1}{M}$. For the case where $\frac{K-1}{M}$ is not an integer, let x be the largest integer less than K such that $\frac{x-1}{M}$ is an integer. Now, we ignore the last $K-x$ users and bound the sum DoF for the remaining users by $\frac{x(M-1)+1}{M}$ to show that $\eta_F(K,M) \leq \frac{x(M-1)+1}{M} + (K-x)$, and hence,

$$\begin{aligned} \eta_F(K,M) &\leq \frac{x(M-1)+1}{M} + (K-x) \\ &= \frac{K(M-1)+1}{M} + \frac{K-x}{M} \\ &\leq \frac{K(M-1)+1}{M} + 1 \end{aligned} \tag{22}$$

■

Together with the achievability result in [8], the statement in Theorem 2 implies the following corollary.

Corollary 2:

$$\tau_F(2) = \frac{1}{2} \tag{23}$$

The characterization of $\tau_F(M)$ for values of $M > 2$ remains an open question, as Theorem 2 is only an upper bound. Moreover, the following result shows that the upper bound in Theorem 2 is loose for $M = 3$.

Theorem 3:

$$\tau_F(3) \leq \frac{5}{8} \tag{24}$$

Proof: We prove the statement by induction, and in order to do so, we use Lemma 2 to provide the basis, and for the induction step, we use Lemma 3 together with the following lemma.

Lemma 4: For $M = 3$, If $\exists \mathcal{A} \subset [K]$ such that $|\mathcal{A}| = n$, and $\frac{K+1}{4} \leq n < K$, $|C_{\mathcal{A}}| \leq n + \frac{K+1}{4} + 1$, then $\exists \mathcal{B} \subset [K]$ such that $|\mathcal{B}| = n+1$, $|C_{\mathcal{B}}| \leq n + \frac{K+1}{4} + 2$.

The proof of the above Lemma follows in a similar fashion to that of Lemma 3. Let $x = n + \frac{K+1}{4} + 1$. We only consider the case where $K > x + 1$, as otherwise, the proof is trivial. We first assume the following,

$$3(K - |C_{\mathcal{A}}|) < (K-n)\left(n + \frac{K+1}{4} + 3 - |C_{\mathcal{A}}|\right) \tag{25}$$

Now, it follows that,

$$\begin{aligned} \sum_{i \in [K], i \notin \mathcal{A}} |C_{\{i\}}| &\leq M(K - |C_{\mathcal{A}}|) \\ &< (K-n)\left(n + \frac{K+1}{4} + 3 - |C_{\mathcal{A}}|\right), \end{aligned} \tag{26}$$

and hence,

$$\exists i \in [K] \backslash \mathcal{A} : |C_{\{i\}} \backslash C_{\mathcal{A}}| \leq n + \frac{K+1}{4} + 2 - |C_{\mathcal{A}}|, \tag{27}$$

and then the set $\mathcal{B} = \mathcal{A} \cup \{i\}$ satisfies the statement of the lemma. Finally, we need to show that (25) is true. For the case where $|C_{\mathcal{A}}| = x$,

$$\begin{aligned} 3x &= \frac{3K}{4} + \frac{15}{4} + 3n \\ &= (2n+K) + (n - \frac{K}{4} + \frac{15}{4}) \\ &> 2n + K, \end{aligned} \tag{28}$$

and hence, $3(K-x) < 2(K-n)$, which implies (25) for the case where $|C_{\mathcal{A}}| = x$. Moreover, we note that each decrement of $|C_{\mathcal{A}}|$ increases the left hand side of (25) by 3 and the right hand side by $(K-n)$, and we know that,

$$\begin{aligned} K &> x + 1 \\ &= n + \frac{K+1}{4} + 2 \\ &\geq n + 2, \end{aligned} \tag{29}$$

and hence, $K - n \geq 3$, so there is no loss of generality in assuming that $|C_{\mathcal{A}}| = x$ in the proof of (25), and the statement of Lemma 4 holds.

Now, we show that $\tau_F(3) = \lim_{K \to \infty} \frac{\eta_F(K,3)}{K} \leq \frac{5}{8}$. It suffices to show that $\eta_F(K,3) \leq \frac{5K}{8} + o(K)$ for all values of K such that $\frac{K+1}{4}$ is an even positive integer, and hence, we make that assumption for K. Define the following,

$$x_1 = \frac{K+1}{4} \tag{30}$$

$$x_2 = \frac{K-7}{8} \tag{31}$$

$$x_3 = 2x_1 + 1 + x_2 \tag{32}$$

Now, we note that,

$$x_3 = K - (x_1 + x_2), \tag{33}$$

and by induction, it follows from lemmas 2 and 3 that $\exists \mathcal{S}_1 \subset [K]$, $|\mathcal{S}_1| = x_1$, $|C_{\mathcal{S}_1}| \leq 2x_1 + 1$. We now apply induction again with the set $\mathcal{S}_1$ as a basis and use Lemma 4 for the induction step to show that $\exists \mathcal{S}_2 \subset [K]$, $|\mathcal{S}_2| = x_1 + x_2$, $|C_{\mathcal{S}_2}| \leq x_3 = K - |\mathcal{S}_2|$, and hence, we get the following upper bound using Corollary 1,

$$\begin{aligned} \eta_F(K,3) &\leq x_3 \\ &= \frac{5(K+1)}{8}, \end{aligned} \tag{34}$$

from which (24) holds. ■

We note that all the DoF upper bounding proofs used so far employ Corollary 1. In [1], we showed that under the hypothesis that the upper bound in Corollary 1 is tight for any K-user fully connected interference channel with a cooperation order constraint M, then scalable DoF cooperation gains are achievable for any value of $M \geq 3$.

Hence, a solution to the general problem necessitates the discovery of either new upper bounding techniques or new coding schemes. However, as we show next, scalable DoF cooperation gains are possible when the assumption of full connectivity is relaxed.

V. Locally Connected Interference Channels

In Section II-A, we defined the locally connected channel model as a function of the number of dominant interferers per receiver L, by connecting each transmitter to $\lfloor \frac{L}{2} \rfloor$ preceding receivers and $\lceil \frac{L}{2} \rceil$ succeeding receivers, and in Section II-C, we illustrated an equivalent model in terms of the asymptotic per user DoF $\tau_L(M)$. In the equivalent model, each transmitter is connected to L succeeding receivers. More precisely, we consider the following channel model,

$$H_{ij} \text{ is not identically } 0 \text{ if and only if } j \in [i, i+1, \ldots, i+L] \tag{35}$$

and all non-zero channel coefficients are generic.

A. Prior Work

In [12], the special case of Wyner's asymmetric model ($L = 1$) was considered, and the spiral message assignment strategy mentioned in Section IV-A was fixed, each message is assigned to its own transmitter as well as $M-1$ following transmitters. The asymptotic per user DoF was then characterized as $\frac{M}{M+1}$. This shows for our problem that,

$$\tau_1(M) \geq \frac{M}{M+1} \tag{36}$$

In [26, Remark 2], a message assignment strategy was described to enable the achievability of an asymptotic per user DoF as high as $\frac{2M-1}{2M}$, it can be easily verified that this is indeed true, and hence, we know that,

$$\tau_1(M) \geq \frac{2M-1}{2M} \tag{37}$$

The main difference in the strategy described in [26, Remark 2] from the spiral message assignment strategy considered in [12], is that unlike the spiral strategy, messages are assigned to transmitters in an asymmetric fashion, where we say that a message assignment is symmetric if and only if for all $j, i \in [K], j \neq i$, the transmit set $\mathcal{T}_j$ is obtained by replacing the index i with the index j for elements of the transmit set $\mathcal{T}_i$.

We show that both the message assignment startegy analyzed in [12] and the one suggested in [26] are suboptimal for $L = 1$, and the value of $\tau_1(M)$ is in fact strictly higher than the bounds in (36) and (37). The key idea enabling our result is that each message need not to be available at the transmitter carrying its own index. We start by illustrating a simple example for the case of no cooperation ($M = 1$) that highlights the idea behind our scheme.

B. Example: $M = L = 1$

Let W_1, W_3, be available at X_1, X_2, respectively, and deactivate both the second receiver Y_2 and the third transmitter X_3, then it is easily seen that messages W_1 and W_3 can be received without interfering signals at

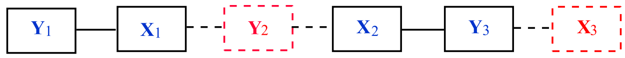

Fig. 2: Achieving $2/3$ per user DoF for $M = L = 1$. Each transmitter is carrying a message for the receiver connected to it by a solid line. Figure showing only signals corresponding to the first 3 users in a general $K-$user network. Signals in dashed boxes are deactivated. Note that the deactivation of X_3 splits this part of the network from the rest.

their corresponding receivers. Moreover, the deactivation of X_3 splits this part of the network from the rest. i.e., the same scheme can be repeated by assigning W_4, W_6, to X_4, X_5, respectively, and so on. Thus, 2 degrees of freedom can be achieved for each set of 3 users, thereby, achieving an asymptotic per user DoF of $\frac{2}{3}$. The described message assignment is depicted in Figure 2. It is evident now that a constraint that is only a function of the load on the *backhaul* link may lead to a discovery of better message assignments than the one considered in [12]. In the following section, we show that the optimal message assignment strategy under the cooperation order constraint (3) is different from the spiral strategy.

C. Achieving Scalable DoF Cooperation Gains

In this Section, we specialize the scheme introduced for multiple-antenna transmitters in [27, Section IV] to our setting. We consider a simple linear precoding coding scheme, where each message is assigned to a set of transmitters with successive indices, and a zero-forcing transmit beamforming strategy is employed. The transmit signal at the j^{th} transmitter is given by,

$$X_j = \sum_{i:j \in \mathcal{T}_i} X_{j,i} \tag{38}$$

where $X_{j,i}$ depends only on message W_i.

Using simple zero-forcing transmit beams with a fractional reuse scheme that activates only a subset of transmitters and receivers in each channel use, we extend the example in Section V-B to achieve scalable DoF cooperation gains for any value of $M > \frac{L}{2}$.

Theorem 4:

$$\tau_L(M) \geq \max\left\{\frac{1}{2}, \frac{2M}{2M+L}\right\}, \forall M \in \mathbf{Z}^+ \tag{39}$$

Proof:

Showing that $\tau_L(M) \geq \frac{1}{2}, \forall M \geq 1$ follows by a straightforward extension of the asymtotic interference alignment sheme of [8], and hence, it suffices to show that $\tau_L(M) \geq \frac{2M}{2M+L}$.

We treat the network as a set of clusters, each consisting of consecutive $2M + L$ transceivers. The last L transmitters of each cluster are deactivated to eliminate *inter-cluster* interference. It then suffices to show that $2M$ DoF can be achieved in each cluster. Without loss of generality, consider the cluster with users of indices in the

set $[2M+L]$. We define the following subsets of $[2M+L]$,

$$\begin{aligned}\mathcal{S}_1 &= [M]\\ \mathcal{S}_2 &= \{L+M+1, L+M+2, \ldots, L+2M\}\end{aligned}$$

We next show that each user in $\mathcal{S}_1 \cup \mathcal{S}_2$ achieves one degree of freedom while messages $\{W_{M+1}, W_{M+2}, \ldots, W_{L+M}\}$ are not transmitted. In the proposed scheme, users in the set $\mathcal{S}_1$ are served by transmitters in the set $\{X_1, X_2, \ldots, X_M\}$ and users in the set $\mathcal{S}_2$ are served by transmitters in the set $\{X_{M+1}, X_{M+2}, \ldots, X_{2M}\}$. Let the message assignments be as follows.

$$\mathcal{T}_i = \begin{cases}\{i, i+1, \ldots, M\}, & \forall i \in \mathcal{S}_1\\ \{i-L, i-L-1, \ldots, M+1\}, & \forall i \in \mathcal{S}_2\end{cases}$$

Now, we note that messages with indices in $\mathcal{S}_1$ are not available outside transmitters with indices in $[M]$, and hence, do not cause interference at receivers with indices in $\mathcal{S}_2$. Also, messages with indices in $\mathcal{S}_2$ are not available at transmitters with indices in $[M]$, and hence, do not cause interference at receivers with indices in $\mathcal{S}_1$.

In order to complete the proof by showing that each user in $\mathcal{S}_1 \cup \mathcal{S}_2$ achieves one degree of freedom, we next show that transmissions corresponding to messages with indices in $\mathcal{S}_1(\mathcal{S}_2)$ do not cause interference at receivers with indices in the same set. To avoid redundancy, we only describe in detail the design of transmit beams for message W_1 to cancel its interference at all receivers in $\mathcal{S}_1$ except its own receiver. First, the encoding of W_1 into $X_{1,1}$ at the first transmitter is done in a way that is oblivious to the existence of other receivers in the network except the first receiver, and a capacity achieving code for the point-to-point link $H_{1,1}$ is used. We then design $X_{2,1}$ at the second transmitter to cancel the interference caused by W_1 at the second receiver, i.e.,

$$X_{2,1} = -\frac{H_{2,1}}{H_{2,2}} X_{1,1} \tag{40}$$

Similarly, the transmit beam $X_{3,1}$ is then designed to cancel the interference caused by W_1 at the third receiver. The transmit beams $X_{i,1}, i \in \{2, 3, \ldots, M\}$ are successively designed with respect to order of the index i such that the received signal due to $X_{i,1}$ at the i^{th} receiver cancels the interference caused by W_1.

In general, the availability of channel state information at the transmitters allows a design for the transmit beams for message W_i that delivers it to the i^{th} receiver with a capacity achieving point-to-point code and simultaneously cancels its effect at receivers with indices in the set $\mathcal{C}_i$, where,

$$\mathcal{C}_i = \begin{cases}\{i+1, i+2, \ldots, M\}, & \forall i \in \mathcal{S}_1\\ \{i-1, i-2, \ldots, L+M+1\}, & \forall i \in \mathcal{S}_2\end{cases}$$

Note that both $\mathcal{C}_M$ and $\mathcal{C}_{L+M+1}$ equal the empty set, because both W_M and W_{L+M+1} do not contribute to interfering signals at receivers in the set $Y_{\mathcal{S}_1} \cup Y_{\mathcal{S}_2}$. We conclude that each receiver with index in the set $\mathcal{S}_1 \cup \mathcal{S}_2$ suffers only from Gaussian noise, thereby enjoying one degree of freedom.

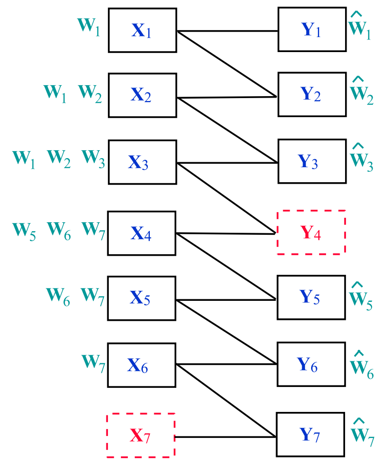

Fig. 3: Figure showing the assignment of messages in the proof of Theorem 4 for the case where $M = 3$ and $L = 1$. Only signals corresponding to the first cluster are shown. Signals in dashed red boxes are deactivated. Note that the last transmit signal is deactivated to eliminate inter-cluster interference. Also, W_4 is not transmitted, while each other message with indices in $\{1, 2, \ldots, 7\}$ has one degree of freedom.

■

We refer the reader to Figure 3 for an illustration of the above described coding scheme. We note that in the above coding scheme, some messages are not being transmitted in order to allow for interference-free communication for the remaining messages. It is worth noting that this can be done while maintaining fairness in the allocation of the available DoF over all users through *fractional reuse* in a system where multiple sessions of communication take place, and different sets of receivers are deactivated in different sessions. e.g., in different time slots or different sub-carriers (in an OFDM system).

D. Useful Message Assignments and Optimality of Local Cooperation

In order to find an upper bound on the per user DoF $\tau_L(M)$, we have to consider all possible message assignment strategies satisfying the cooperation order constraint (3). In this section, we characterize necessary conditions for the optimal message assignment. The constraints we provide for transmit sets are governed by the connectivity pattern of the channel. For example, for the case where $M = 1$, any assignment of message W_i to a transmitter that is not connected to Y_i is not *useful*, i.e., the rate of transmitting message W_i has to be zero for those assignments.

We now introduce a graph theoretic representation that simplifies the presentation of the necessary conditions on useful message assignments. For message W_i, and a fixed transmit set $\mathcal{T}_i$, we construct the following graph $G_{W_i,\mathcal{T}_i}$ that has $[K]$ as its set of vertices, and an edge exists between any given pair of vertices $x, y \in [K]$ if and only if:

- $x, y \in \mathcal{T}_i$.
- $|x - y| \leq L$.

Vertices corresponding to transmitters connected to Y_i are given a special mark, i.e., vertices with labels in the set $\{i, i-1, \ldots, i-L\}$ are marked for the considered channel model.

We now have the following statement.

Lemma 5: For any $k \in \mathcal{T}_i$ such that the vertex k in $G_{W_i,\mathcal{T}_i}$ is not connected to a marked vertex, removing k from $\mathcal{T}_i$ does not decrease the sum rate.

Proof: Let $\mathcal{S}$ denote the set of indices of vertices in a component with no marked vertices. We need to show that removing any transmitter in $\mathcal{S}$ from $\mathcal{T}_i$ does not decrease the sum rate. Let $\mathcal{S}'$ be the set of indices of received signals that are connected to at least one transmitter with an index in $\mathcal{S}$. To prove the lemma, we consider two scenarios, where we add a *tilde* over symbols denoting rates and signals belonging to the second scenario. For the first scenario, W_i is made available at transmitters in $\mathcal{S}$. Let Q be a random variable that is independent of all messages and has the same distribution as W_i, then for the second scenario, W_i is not available at transmitters in $\mathcal{S}$, and a realization q of Q is generated and given to all nodes in $\tilde{X}_{\mathcal{S}} \cup \tilde{Y}_{\mathcal{S}'}$ before communication starts. Moreover, the given realization $Q = q$ contributes to the encoding of $\tilde{X}_{\mathcal{S}}$ in the same fashion as a message $W_i = q$ contributes to $X_{\mathcal{S}}$. Assuming a reliable communication scheme for the first scenario that uses a large block length n, the following argument shows that the achievable sum rate is also achievable after removing W_i from the designated transmitters. Therefore, proving that removing any transmitter in $\mathcal{S}$ from $\mathcal{T}_i$ does not decrease the sum rate.

$$
\begin{aligned}
n\sum_j R_j &= \sum_j H(W_j) \\
&\overset{(a)}{\leq} \sum_j I(W_j;Y_j) + o(n) \\
&= \sum_{j\in\mathcal{S}'^c} I(W_j,Y_j) + \sum_{j\in\mathcal{S}'} I(W_j;Y_j) + o(n) \\
&\overset{(b)}{=} \sum_{j\in\mathcal{S}'^c} I(W_j,\tilde{Y}_j) + \sum_{j\in\mathcal{S}'} I(W_j;Y_j) + o(n) \\
&\leq \sum_{j\in\mathcal{S}'^c} I(W_j,\tilde{Y}_j) + \sum_{j\in\mathcal{S}'} I(W_j;Y_j|W_i) + o(n) \\
&\overset{(c)}{=} \sum_{j\in\mathcal{S}'^c} I(W_j,\tilde{Y}_j) + \sum_{j\in\mathcal{S}'} I(W_j;\tilde{Y}_j|Q) + o(n) \\
&\leq n\sum_j \tilde{R}_j + o(n)
\end{aligned}
$$

where (a) follows from Fano's inequality, (b) follows as the difference between the two scenarios lies in the encoding of $X_{\mathcal{S}}$ which affects only $Y_{\mathcal{S}'}$, and (c) follows as there are no transmitters outside $X_{\mathcal{S}}$ that are carrying W_i and connected to $Y_{\mathcal{S}'}$, and $Y_i \notin Y_{\mathcal{S}'}$. ■

We call a message assignment *useful* if no element in it can be removed without decreasing the sum rate. The following corollary to the above lemma characterizes a necessary condition for any message assignment satisfying the cooperation order constraint in (3) to be useful.

Corollary 3: Let $\mathcal{T}_i$ be a useful message assignment and $|\mathcal{T}_i| \leq M$, then $\forall k \in [K], k \in \mathcal{T}_i$ only if the vertex k in $G_{W_i,\mathcal{T}_i}$ lies at a distance that is less than or equal $M-1$ from a marked vertex.

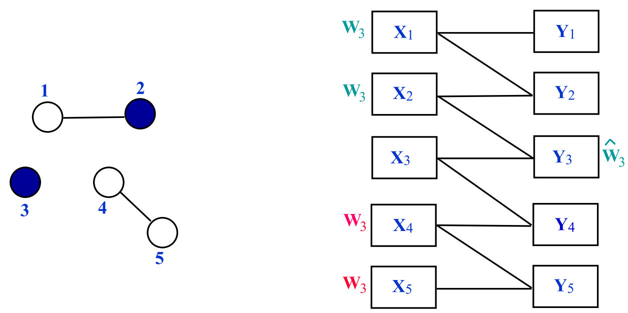

Fig. 4: Figure showing the construction of $G_{W_3,\mathcal{T}_3}$ in a $5-$user channel with $L=1$. Marked vertices are represented with filled circles. W_3 can be removed at both X_4 and X_5 without decreasing the sum rate, as the corresponding vertices lie in a component that does not contain a marked vertex.

Note that in the considered channel model, the above result implies that $\mathcal{T}_i \subseteq \{i-ML, i-ML+1, \ldots, i+(M-1)L\}$, from which we obtain the following result.

Theorem 5: Local cooperation is optimal for locally connected channels,

$$\tau_L^{(\text{loc})}(M) = \tau_L(M), \forall M, L \in \mathbf{Z}^+ \tag{41}$$

And so we note that even though local cooperation does not achieve a scalable DoF gain for the fully connected channel, not only does it achieve a scalable gain when the connectivity assumption is relaxed to local connectivity, but the confinement to local cooperation no longer results in a loss in the available DoF.

E. DoF Upper Bounds

In this section, we prove upper bounds on $\tau_1(M)$ and $\tau_L(1)$ that establishes the tightness of the lower bound in Theorem 4 for the special cases where either $L=1$ or $M=1$. First, in order to assess the optimality of the coding scheme introduced in Section V-C for arbitrary values of the system parameters, we prove a general upper bound for a class of coding schemes that only employs a zero-forcing transmit beam-forming strategy.

1) ZF Transmit Beam-Forming: Consider only coding schemes with transmit signals of the form (38) and each message is either not transmitted or allocated one degree of freedom. More precisely, let $\tilde{Y}_j = Y_j - Z_j, \forall j \in [K]$, then in addition to the constraint in (38), the differential entropy $h(\tilde{Y}_j|W_j)$ is either 0 or equals $h(\tilde{Y}_j)$ for all $j \in [K]$. Note that $h(\tilde{Y}_j|W_j) = 0$ for the case where user j enjoys interference-free communication and $h(\tilde{Y}_j|W_j) = h(\tilde{Y}_j)$ for the other case where W_j is not transmitted. We say that the j^{th} receiver is *active* if and only if $h(\tilde{Y}_j|W_j) \neq h(Y_j)$.

Let $\tau_L^{(\text{zf})}(M)$ denote the asymptotic characterization of the per user DoF number under the restriction to the above described class of coding schemes. In Theorem 6 below, we show that the coding scheme in the proof of Theorem 4 achieves the optimal value of $\tau_L^{(\text{zf})}(M)$. We first prove Lemma 6 that bounds the number of receivers at which the interference of a given message can be canceled.

For a set $\mathcal{S} \subseteq [K]$, let $\mathcal{V}_\mathcal{S}$ be the set of indices for active receivers connected to transmitters with indices in $\mathcal{S}$. More precisely, $\mathcal{V}_\mathcal{S} = \{j : h(\tilde{Y}_j|W_j) = 0, \mathcal{S} \cap \{j, j-1, \ldots, j-L\} \neq \phi\}$, where ϕ is the empty set. To obtain the

following results, we assume that for each transmitter in $\mathcal{T}_i$, message W_i contributes to the transmit signal of this transmitter. i.e., $\forall j \in \mathcal{T}_i, I(W_i, X_j) > 0$. Note that this assumption does not introduce a loss in generality, because otherwise the transmitter can be removed from $\mathcal{T}_i$. We need the following lemma for the proof of the upper bound on $\tau_L^{(\mathrm{zf})}(M)$ in Theorem 6.

Lemma 6:

$$|\mathcal{V}_{\mathcal{T}_i}| \leq |\mathcal{T}_i| \tag{42}$$

Proof: We only consider the non-trivial case where $\mathcal{T}_i \neq \phi$. For each receiver $j \in \mathcal{V}_{\mathcal{T}_i}$, there exists a transmit signal $X_{k,i}$, $k \in [K]$ such that conditioned on all other transmit signals, the received signal Y_j is correlated with the message W_i. More precisely, $I\left(W_i; Y_j | \{X_{v,i}, v \in [K], v \neq k\}\right) > 0$. Now, since we impose the constraint $I(W_i; Y_j) = 0, \forall j \in \mathcal{V}_{\mathcal{T}_i}$, the interference seen at all receivers in $\mathcal{V}_{\mathcal{T}_i}$ has to be canceled. Finally, since the probability of a zero Lebesgue measure set of channel realizations is zero, the $|\mathcal{T}_i|$ transmit signals carrying W_i cannot be designed to cancel W_i at more than $|\mathcal{T}_i| - 1$ receivers for almost all channel realizations. ■

Theorem 6:

$$\tau_L^{(\mathrm{zf})}(M) = \frac{2M}{2M+L} \tag{43}$$

Proof: The proof of the lower bound is the same as the proof of Theorem 4 for the case where $\frac{2M}{2M+L} > \frac{1}{2}$. It then suffices to show that $\tau_L^{(\mathrm{zf})}(M) \leq \frac{2M}{2M+L}$. In order to prove the upper bound, we show that the sum degree of freedom in each set $\mathcal{S} \subseteq [K]$ of consecutive $2M+L$ users is bounded by $2M$. Assume otherwise, then there is an active user in $\mathcal{S}$ that lies in the middle of a subset of $2M+1$ active users in $\mathcal{S}$. More precisely, $\exists i \in \mathcal{S} : |\mathcal{T}_i| > 0, |\{j : j < i, j \in \mathcal{S}, h(\tilde{Y}_j|W_j) = 0\}| \geq M, |\{j : j > i, j \in \mathcal{S}, h(\tilde{Y}_j|W_j) = 0\}| \geq M$. Let $s_{min} = \min_s\{s : s \in \mathcal{S}\}$ and $s_{max} = \max_s\{s : s \in \mathcal{S}\}$, then if $\exists s \in \mathcal{T}_i : s \in \{s_{min}, s_{min}-1, \ldots, s_{min}-L\}$, it follows that $\mathcal{V}_{\mathcal{T}_i} \supseteq \{j : j < i, j \in \mathcal{S}, h(\tilde{Y}_j|W_j) = 0\} \cup \{i\}$. Hence $|\mathcal{V}_{\mathcal{T}_i}| \geq M+1$, which contradicts (42), as $|\mathcal{T}_i| \leq M$. Also, if $\exists s \in \mathcal{T}_i : s \in \{s_{max}, s_{max}-1, \ldots, s_{max}-L\}$, then $\mathcal{V}_{\mathcal{T}_i} \supseteq \{j : j > i, j \in \mathcal{S}, h(\tilde{Y}_j|W_j) = 0\} \cup \{i\}$. Hence $|\mathcal{V}_{\mathcal{T}_i}| \geq M+1$, which again contradicts (42). Finally, for the remaining case, by applying Lemma 5, we know that $\mathcal{T}_i$ does not contain a transmitter with an index that is less than s_{min} or greater than s_{max}, hence, at least $L+|\mathcal{T}_i|$ receivers in $\mathcal{S}$ are connected to one or more transmitters in $\mathcal{T}_i$, and since $\mathcal{S}$ has at least $2M+1$ active receivers, then any subset of $L+|\mathcal{T}_i|$ receivers in $\mathcal{S}$ has to have at least $2M+1-((2M+L)-(L+|\mathcal{T}_i|)) = |\mathcal{T}_i|+1$ active receivers, and the statement is proved by reaching a contradiction to (42) in the last case. ■

2) Wyner's Asymmetric Model: Now, we consider the special case of $L=1$, and prove that the lower bound stated in Theorem 4 is tight in that case. More precisely, we show the following,

Theorem 7: The asymptotic per user DoF for Wyner's asymmetric model with CoMP transmission is,

$$\tau_1(M) = \frac{2M}{2M+1}, \forall M \in \mathbf{Z}^+ \tag{44}$$

Proof: The lower bound follows from Theorem 4. In order to prove the converse, we use Lemma 1 with a set $\mathcal{A}$ of size $K\frac{2M}{2M+1} + o(K)$. We also prove the upper bound for the channel after removing the first M transmitters $\left(X_{[M]}\right)$, while noting that this will be a valid bound on $\tau_1(M)$ since the number of removed transmitters is $o(K)$.

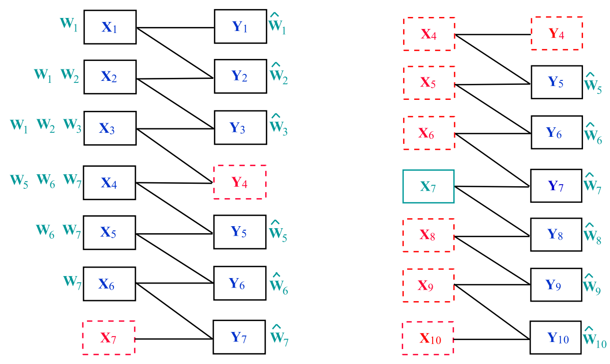

Fig. 5: Figure illustrating the proof of Theorem 7 for $M = 3$, $\tau(3) = \frac{6}{7}$. In (a), the message assignments in the first cluster for the proposed coding scheme are illustrated. Note that both X_7 and Y_4 are deactivated. In (b), an illustration of the upper bound is shown. The messages W_4 and W_{11} cannot be available at X_7, hence it can be reconstructed from $W_{\mathcal{A}}$. All transmit signals shown in figure can be reconstructed from X_7 and noise free versions of $\{Y_5, \ldots, Y_{10}\}$.

Inspired by the coding scheme in the proof of Theorem 4, we define the set $\mathcal{A}$ as the set of receivers that are *active* in the coding scheme. i.e., the complement set $\bar{\mathcal{A}} = \{i : i \in [K], i = (2M+1)(j-1)+M+1, j \in \mathbf{Z}^+\}$. We know from Corollary 3 that messages belonging to the set $W_{\bar{\mathcal{A}}}$ do not contribute to transmit signals with indices that are multiples of $2M+1$, i.e., $i \notin U_{\mathcal{A}}$ for all $i \in [K]$ that is a multiple of $2M+1$. More precisely, let the set $\mathcal{S}$ be defined as follows:

$$\mathcal{S} = \{i : i \in [K], i \text{ is a multiple of } 2M+1\}$$

then $\mathcal{S} \subseteq \bar{U}_{\mathcal{A}}$. In particular, $X_{\mathcal{S}} \subseteq X_{\bar{U}_{\mathcal{A}}}$, and hence it suffices to show the existence of a function f such that $f\left(Y_{\mathcal{A}}, Z_{\mathcal{A}}, X_{\mathcal{S}}\right) = X_{\bar{\mathcal{S}}} \backslash X_{[M]}$.

In what follows we show how to reconstruct the signals in the set $\{X_{M+1}, X_{M+2}, \ldots, X_{2M}\} \cup \{X_{2M+2}, X_{2M+3}, \ldots, X_{3M+1}\}$. Then it will be clear by symmetry how to reconstruct the rest of transmit signals in the set $X_{\bar{\mathcal{S}}} \backslash X_{[M]}$. Since $X_{2M+1} \in X_{\mathcal{S}}$, and a noise free version of Y_{2M+1} is also given, X_{2M} can be reconstructed. Now, with the knowledge of X_{2M}, Y_{2M}, and Z_{2M}, we can reconstruct X_{2M-1}, and so by iterative processing, all transmit signals in the set $\{X_{M+1}, X_{M+2}, \ldots, X_{2M}\}$ can be reconstructed. In a similar fashion, given X_{2M+1}, Y_{2M+2}, and Z_{2M+2}, the signal X_{2M+2} can be reconstructed. Then with a noise free version of Y_{2M+3}, we can reconstruct X_{2M+3}, and we can proceed along this path to reconstruct all transmit signals in the set $\{X_{2M+2}, X_{2M+3}, \ldots, X_{3M+1}\}$. This proves the existence of the function f defined above, and so by Lemma 1 we obtain the converse of Theorem 7. ■

In Figure 5 (b), we illustrate how the proof works for the case where $M = 3$. Note that the missing received signals $\{Y_4, Y_{11}, \ldots\}$ in the upper bound proof correspond to the inactive receivers in the coding scheme.

3) No Cooperation: We note that even for the case of *no cooperation*, an asymptotic per user DoF of more than $\frac{1}{2}$ per user DoF is achievable, i.e., $\tau_1(1) = \frac{2}{3}$. Also, it is straightforward to see that the interference alignment scheme can be generalized to show that $\tau_L(1) \geq \frac{1}{2}$ for any locally connected channel with parameter L. The next theorem generalizes the upper bound in [7] for locally connected channels, where each message can be available at one transmitter that is not necessarily the transmitter carrying its own index. In particular, we show that $\tau_L(1) > \frac{1}{2}$ only if $L = 1$.

Theorem 8: Without cooperation ($M = 1$), the asymptotic per user DoF of locally connected channels is given by

$$\tau_L(1) = \begin{cases} \frac{2}{3}, & \text{if} \quad L = 1 \\ \frac{1}{2}, & \text{if} \quad L \geq 2 \end{cases}$$

Proof: The case where $L = 1$ is a special case of the result in Theorem 7. The lower bound for the case where $L \geq 2$ follows by assigning each message to the transmitter with the same index, and a simple extension of the asymptotic interference alignment scheme of [8], and hence, it suffices to show that,

$$\tau_L(1) \leq \frac{1}{2}, \forall L \geq 2 \tag{45}$$

In particular, we establish the stronger statement,

$$\eta_L(K, 1) \leq \frac{K}{2}, \forall K, \forall L \geq 2 \tag{46}$$

We prove the statement by induction. First, we prove the following lemma which serves as a building block for proving both the basis and induction step. We define $\mathcal{R}_i$ as the set of indices of received signals that are connected to transmitter X_i, i.e., $\mathcal{R}_i = \{i, i+1, \ldots, i+L\}$. Note that as we are considering the case of *no cooperation*, hence, $\mathcal{T}_i$ contains only one element. Let d_i denote the available DoF for the communication of message W_i.

Lemma 7: If $\mathcal{T}_i = \{X_j\}$ then $d_i + d_s \leq 1, \forall s \in \mathcal{R}_j, s \neq i$.

Proof: We assume that all messages other than W_i and W_s are deterministic, and then apply Lemma 1 with the set $\mathcal{A} = \{s\}$; Then the bound follows. ■

The basis to the induction step follows directly from the above lemma.

Lemma 8: For $M = 1$, $\forall L, d_1 + d_2 \leq 1$

Proof: The proof follows from the above lemma and the fact that all transmitters connected to Y_1 are also connected to Y_2. ■

Finally, the following lemma completes the proof through the induction step. Let B_k be a boolean variable that is true if and only if the following is true

- $\frac{\sum_{i=1}^{k} d_i}{k} \leq \frac{1}{2}$
- $d_{k-1} + d_k \leq 1$

Lemma 9: For $L \geq 2$, $k \geq 2$, if B_k is true, then either B_{k+1} or B_{k+2} is true.

Proof: The proof follows from Lemma 7, and the fact that W_{k+1} is available at a transmitter connected to Y_{k+1}, which is either connected to Y_{k+2}, or to both Y_k and Y_{k-1}. ■

To complete the proof, we note that for $L \geq 2$, all the transmitters connected to the last receiver Y_K are also connected to Y_{K-1}. Hence if B_{K-1} is true, then so is B_K. ∎

VI. Discussion

There are two design parameters in the considered problem, the message assignment strategy satisfying the cooperation order constraint, and the design of transmit beams. We characterized the asymptotic per user DoF when one of the design parameters is restricted to a special choice, i.e., restricting message assignment strategies by a local cooperation constraint or restricting the design of transmit beams to zero-forcing transmit beams. The restriction of one of the design parameters can significantly simplify the problem because of the inter-dependence of the two design parameters. On one hand, the achievable scheme is enabled by the choice of the message assignment strategy, and on the other hand, the assignment of messages to transmitters is governed by the technique followed in the design of transmit beams, e.g. zero-forcing transmit beamforming or interference alignment. In the following, we discuss each of the design parameters.

A. Message Assignment Strategy

The assignment of each message to more than one transmitter (CoMP transmission) creates a virtual MISO network. A real MISO network, where multiple dedicated antennas are assigned to the transmission of each message, differs from the created virtual one in two aspects. First, in a CoMP transmission setting, the same transmit antenna can carry more than one message. Second, for locally connected channels, the number of receivers at which a message causes undesired interference depends on the number of transmit antennas carrying the message.

For fully connected channels, the number of receivers at which a message causes undesired interference is the same regardless of the size of the transmit set as long as it is non-empty. The only aspect that governs the assignment of messages to transmitters is the pattern of overlap between transmit sets corresponding to different messages. It is expected that the larger the sizes of the intersections between sets of messages carried by different transmit antennas, the lower the available DoF. For the spiral assignments of messages considered in [11], $|\mathcal{T}_i \cap \mathcal{T}_{i+1}| = M - 1$, and the same value holds for the size of the intersection between sets of messages carried by successive transmitters. In general, local cooperation implies large intersections between sets of messages carried by different transmitters, and hence, the negative conclusion we reached for $\tau_F^{(\text{loc})}(M)$.

For the case where we are restricted to zero-forcing transmit beamforming as in Section V-C, the number of receivers at which each message causes undesired interference governs the choice of transmit sets, and hence, we saw that for locally connected channels, the message assignment strategy illustrated in Theorem 4 selects transmit sets that consist of successive transmitters, to minimize the number of receivers at which each message should be canceled. This strategy is optimal under the restriction to zero-forcing transmit beamforming schemes.

B. Design of Transmit Beams

While it was shown in [11] that CoMP transmission accompanied by both zero-forcing transmit beams and asymptotic interference alignment can achieve a DoF cooperation gain beyond what can be achieved using only

transmit zero-forcing, this is not obvious for locally connected channels. Unlike in the fully connected channel, the addition of a transmitter to a transmit set in a locally connected channel may result in an increase in the number of receivers at which the message causes undesired interference.

We note that unlike asymptotic interference alignment scheme, the zero-forcing transmit beamforming scheme illustrated in Section V-C does not need symbol extensions, since it achieves the stated DoF of Theorem 4 in one channel realization. However, we believe that the question of whether asymptotic interference alignment can be used to show an asymptotic per user DoF cooperation gain beyond that achieved through simple zero-forcing transmit beamforming, is closely related to the answer of both questions that remain open after this work, that is, whether $\tau_F(M) > \frac{1}{2}$ for $M > 2$, and whether the lower bound showed in Theorem 4 is tight for locally connected channels.

VII. Conclusions

The DoF gain achieved through CoMP transmission was studied. In particular, it was of interest to know whether the achievable gain scales linearly with K as it goes to infinity, under a cooperation constraint that only limits the number of transmitters at which any message can be available by a cooperation order M. The answer was shown to be negative for the fully connected channel where message assignment strategies satisfy the local cooperation constraint, as well as all possible message assignments for the case where $M = 2$. The problem is still open for fully connected channels and values of $M \geq 3$.

It was shown for locally connected channels where each transmitter is connected to the receiver carrying the same index as well as L neighboring receivers, that the asymptotic per user DoF is lower bounded by $\max\left\{\frac{1}{2}, \frac{2M}{2M+L}\right\}$. The achieving coding scheme is simple as it relies only on zero-forcing transmit beamforming. This lower bound was shown to be tight for the case where $L = 1$. In particular, the characterized asymptotic per user DoF for that case is $\frac{2M}{2M+1}$, and is higher than previous results in [12], and [26].

Insights on the optimal way of assigning messages to transmitters under a cooperation order constraint were revealed. A local cooperation constraint was considered, where each message can only be available at a neighborhood of transmitters whose size does not scale linearly with the number of users. While local cooperation was shown not to achieve a scalable DoF gain for the fully connected channel, it was also shown to be optimal for locally connected channels. Furthermore, light has been shed on the intimate relation between the selection of message assignments and the design of transmit beams. It has been shown that assigning messages to successive transmitters is beneficial for zero-forcing transmit beamforming in locally connected channels as it minimizes the number of receivers at which each message causes undesired interference. However, the same message assignment strategy can be an impediment to other techniques such as asymptotic interference alignment, because the overlap of sets of messages carried by transmit antennas is large for this assignment of messages.

APPENDIX

PROOF OF LEMMA 1

In order to prove the lemma, we show that using a reliable communication scheme with the aid of a signal that is within $o(\log P)$, all the messages can be recovered from the set of received signals $Y_{\mathcal{A}}$. It follows that any achievable degree of freedom for the channel is also achievable for another channel that has only those receivers, thus proving the upper bound.

In any reliable n-block coding scheme,

$$H(W_i|Y_i^n) \leq n\epsilon, \forall i \in [K].$$

Therefore,

$$H(W_{\mathcal{A}}|Y_{\mathcal{A}}^n) \leq \sum_{i\in\mathcal{A}} H(W_i|Y_i^n) \leq n|\mathcal{A}|\epsilon.$$

Now, the sum $\sum_{i\in[K]} R_i = \sum_{i\in\bar{\mathcal{A}}} R_i + \sum_{i\in\mathcal{A}} R_i$ can be bounded as

$$\begin{aligned} n\left(\sum_{i\in\bar{\mathcal{A}}} R_i + \sum_{i\in\mathcal{A}} R_i\right) &= H(W_{\bar{\mathcal{A}}}) + H(W_{\mathcal{A}}) \\ &\leq I\left(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n\right) + I\left(W_{\mathcal{A}}; Y_{\mathcal{A}}^n\right) \\ &\quad +nK\epsilon. \end{aligned} \tag{47}$$

where ϵ can be made arbitrarily small, by choosing n large enough. The two terms on the right hand side of (47) can be bounded as

$$\begin{aligned} I\left(W_{\mathcal{A}}; Y_{\mathcal{A}}^n\right) &= \mathsf{h}\left(Y_{\mathcal{A}}^n\right) - \mathsf{h}\left(Y_{\mathcal{A}}^n|W_{\mathcal{A}}\right) \\ &\leq \sum_{i\in\mathcal{A}}\sum_{t=1}^{n} \mathsf{h}\left(Y_i(t)\right) - \mathsf{h}\left(Y_{\mathcal{A}}^n|W_{\mathcal{A}}\right) \\ &= |\mathcal{A}|n\log P + n(o(\log P)) - \mathsf{h}\left(Y_{\mathcal{A}}^n|W_{\mathcal{A}}\right) \end{aligned}$$

$$\begin{aligned} I\left(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n\right) &\leq I\left(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n, Y_{\mathcal{A}}^n, W_{\mathcal{A}}\right) \\ &= I(W_{\bar{\mathcal{A}}}; Y_{\mathcal{A}}^n|W_{\mathcal{A}}) + I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n) \\ &= \mathsf{h}\left(Y_{\mathcal{A}}^n|W_{\mathcal{A}}\right) - \mathsf{h}\left(Z_{\mathcal{A}}^n\right) + \mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n\right) \\ &\quad -\mathsf{h}\left(Z_{\bar{\mathcal{A}}}^n\right). \end{aligned}$$

Now, we have

$$\begin{aligned} I\left(W_{\mathcal{A}}; Y_{\mathcal{A}}^n\right) + I\left(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n\right) &\leq |\mathcal{A}|n\log P + \mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n\right) \\ &\quad +n(o(\log P)). \end{aligned}$$

Therefore, if we show that

$$\mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n\right) = n(o(\log P)),$$

then from (47), we have the required outer bound. Since $W_{\mathcal{A}}$ contains all the messages carried by transmitters in $X_{\bar{U}_{\mathcal{A}}}$, they determine those input signals for the n channel uses. Therefore,

$$
\begin{aligned}
\mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n | W_{\mathcal{A}}, Y_{\mathcal{A}}^n\right) &= \mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n | W_{\mathcal{A}}, Y_{\mathcal{A}}^n, X_{\bar{U}_{\mathcal{A}}}^n\right) \\
&\leq \mathsf{h}\left(Y_{\bar{\mathcal{A}}}^n | Y_{\mathcal{A}}^n, X_{\bar{U}_{\mathcal{A}}}^n\right) \\
&\leq \sum_{t=1}^{n} \mathsf{h}\left(Y_{\bar{\mathcal{A}}}(t) | Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&\leq \sum_{t=1}^{n} \mathsf{h}\left(Y_{\bar{\mathcal{A}}}(t), Z_{[K]}(t) | Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&= \sum_{t=1}^{n} \mathsf{h}\left(Y_{\bar{\mathcal{A}}}(t), Z_{[K]}(t), Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&\quad -\mathsf{h}\left(Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&\overset{(a)}{=} \sum_{t=1}^{n} \mathsf{h}\left(Z_{[K]}(t), Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&\quad -\mathsf{h}\left(Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&= \sum_{t=1}^{n} \mathsf{h}\left(Z_{[K]}(t) | Y_{\mathcal{A}}(t), X_{\bar{U}_{\mathcal{A}}}(t)\right) \\
&= n(o(\log P))
\end{aligned}
$$

where (a) follows from the existence of the function f by the statement of the Lemma, as given $Y_{\mathcal{A}}, X_{\bar{U}_{\mathcal{A}}}, Z_{\mathcal{A}}$, then $X_{U_{\mathcal{A}}}$ can be recovered, and hence $Y_{\bar{\mathcal{A}}}$, as $Z_{\bar{\mathcal{A}}}$ is given.

Auxiliary Lemma for Large Networks Upper Bounds

Lemma 10: If $K \geq (M-1)(n+1)+1$, $M \geq 2$, and $\exists \mathcal{S} \subseteq [K]$ such that $|\mathcal{S}| \leq (M-1)n+1$, then,

$$M(K-|\mathcal{S}|) < (K-n)\left((M-1)(n+1)+2-|\mathcal{S}|\right) \tag{48}$$

Proof: We first prove the statement for the case where $|\mathcal{S}| = (M-1)n+1$. This directly follows, as,

$$
\begin{aligned}
M(K-|\mathcal{S}|) &= M(K-((M-1)n+1)) \\
&\leq M(K-(n+1)) \\
&< M(K-n) \\
&= (K-n)\left((M-1)(n+1)+2-|\mathcal{S}|\right)
\end{aligned}
\tag{49}
$$

In order to complete the proof, we note that each decrement of $|\mathcal{S}|$ leads to an increase in the left hand side by

M, and in the right hand side by $K-n$, and,

$$\begin{aligned} K-n &\geq (M-1)(n+1)+1-n \\ &= (M-2)n+M \\ &\geq M \end{aligned} \tag{50}$$

■

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